Recall: $G := Mp_2(\mathbb{R}) \ltimes N$

· we define a (right) action of G on holomorphic functions on $\mathcal{Y}$ as follows $(A, j) \in Mp_2(\mathbb{R})$, $n \in N$.

$$F|_{(A,j)}(\tau, z, u) = j(\tau)^{-l} F(A\cdot(\tau, z, u)), \quad l = \dim(\mathring{H}_R).$$

$$N = \mathring{H}_R \times \mathring{H}_R \times i\mathbb{R}$$

$$F|_n(\vartheta) = F(n\vartheta).$$

· obviously: (13.4.4) $D(F)|_n = D(F|_n)$ for $n \in N$.

- prop 13.4:

a) $\widetilde{Th}_k|_{(A,j)} = \widetilde{Th}_k$ if the lattice M is even and $A \in SL_2(\mathbb{Z})$ or if the lattice M is odd and $A \in \Gamma_\theta$.

b) $Th_k|_{(A,j)} = Th_k$ if . . . .
if . . . .

proof: Recall: $Th_k = \{F \in \widetilde{Th}_k \mid D(F) = 0\}$,

a) follows from lem 13.4 and ~~(13.4.4)~~.

(T1) $\widetilde{Th}_k|_{(A,j)}(n\cdot\vartheta) = j^{-l}\widetilde{Th}_k(A\cdot n\cdot\vartheta) = j^{-l}\widetilde{Th}_k(n\cdot A\cdot\vartheta) = j^{-l}\widetilde{Th}_k(A\cdot\vartheta)$
$= \widetilde{Th}_k|_{(A,j)}(\vartheta)$.

(T2) $\widetilde{Th}_k|_{(A,j)}(\vartheta + a\delta) = \widetilde{Th}_k|_{(A,j)}((0,0,a)(\vartheta)) = j^{-l}\widetilde{Th}_k(A(0,0,a)\vartheta)$
$= j^{-l}\widetilde{Th}_k((0,0,a)A\vartheta) = j^{-l}e^{ka}\widetilde{Th}_k(A\vartheta) = e^{ka}\widetilde{Th}_k|_{(A,j)}(\vartheta)$ 逆群

b). it suffices to check that:
$D(\theta_\lambda|_{(T,1)}) = 0$, $D(\theta_\lambda|_{(S,j)}) = 0$.

① by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot(\tau, z; u) = \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, u - \frac{c(z)^2}{2(c\tau+d)}\right)$.

we have $D(\theta_\lambda|_{(T,1)})(\tau, z, u) = D(\theta_\lambda((T,1)(\tau, z, u)))$
$= \underline{D(\theta_\lambda(\tau+1, z, u))} = (D\theta_\lambda)(\tau+1, z, u) = 0$

$\theta_\lambda|_{(T,1)}(\tau, z, u) = j^{-l}\theta_\lambda(\tau+1, z, u)$

$\theta_\lambda(\tau+1, z, u) = \theta_\lambda|_n(\tau, z, u)$.

· need: $\underline{D(j^{-l}e^{2\pi i k(u - \frac{(r-z)^2}{2\tau})})} = 0$ where $r \in \mathring{H}$.

$= \frac{1}{4\pi i}\left(2\partial_u\partial_\tau - \sum_{i=1}^{l}(\partial_{z_i})^2\right)(j^{-l}e^{(\cdots)})$.

$= \cdots\cdots = 0$

$\theta_\lambda|_{(S,j)}(\tau, z, u) = j^{-l}\theta_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right)$

by (13.2.5) $= j^{-l}e^{2\pi k i(u - \frac{|z|^2}{2\tau})}\sum_{r\in M + k^{-1}\lambda} e^{-\pi k i\frac{(r|r)}{\tau} + 2\pi k i(r|\frac{z}{\tau})}$

$$= \sum_{r\in M + k^{-1}\lambda} \underline{j^{-l}e^{2\pi k i(u - |z-r|^2/2\tau)}}$$

Then $D(\theta_\lambda|_{(S,j)}(\tau,z,u)) = 0$.
since S and T generate $SL_2(\mathbb{Z})$ and S and T^2 gen. Γ_θ. #

$$\text{(13.4.5)} \quad \theta_\lambda|_{(\alpha,0,0)} = e^{2\pi i(\alpha|\lambda)}\theta_\lambda \quad \text{for } \alpha \in k^{-1}M^*$$

$$\text{(13.4.6)} \quad \theta_\lambda|_{(0,\alpha,0)} = \theta_{\lambda-k\alpha} \quad \text{for } \alpha \in k^{-1}M^*.$$

proof: since $(\alpha,0,0)(v) = p_\alpha(v) = v + 2\pi i\alpha = 2\pi i(z+\alpha-\tau\Lambda_0+u\delta)$
i.e. $(\alpha,0,0)(\tau,z,u) = (\tau, z+\alpha, u)$.
Then by (13.2.5)
$\theta_\lambda|_{(\alpha,0,0)}(v) = \theta_\lambda(\tau, z+\alpha, u)$

$$= e^{2\pi iku}\sum_{\gamma\in M+k^{-1}\bar{\lambda}} e^{\pi ik\tau(\gamma|\gamma)+2\pi ik(\gamma|z+\alpha)}$$

$$= e^{2\pi iku}\sum_{\gamma\in M+k^{-1}\bar{\lambda}} e^{2\pi ik(\gamma|\alpha)}\cdot e^{\pi ik\tau(\gamma|\gamma)+2\pi ik(\gamma|z)}$$

$$= \sum_{\gamma\in M} e^{2\pi ik\left(\gamma+\frac{\lambda-k\Lambda_0+(\lambda|\Lambda_0)\delta}{k}\,\middle|\,\alpha\right)}$$

$$= e^{2\pi i(\alpha|\lambda)}\sum_{\gamma\in M} e^{2\pi ik(\gamma|\alpha)} \in 2\pi i\mathbb{Z}.$$

Then $\theta_\lambda|_{(\alpha,0,0)} = e^{2\pi i(\alpha|\lambda)}\theta_\lambda$ for $\alpha\in k^{-1}M^*$.

- Take $\lambda\in P_k$, s.t. $(\lambda|\lambda)=0$. Then
$\theta_\lambda|_{(0,\alpha,0)}(v) = \sum_{\beta\in M} e^{(t_\beta(\lambda)|t_\alpha(v))} = \sum_{\beta\in M} e^{(t_\beta(t_{-\alpha}(\lambda))|v)}$

$= \theta_{t_{-\alpha}(\lambda)}(v) = \theta_{\lambda-k\alpha}(v)$ #

Cor 13.4. The function θ_λ (define by (13.2.1)) is characterized among the holomorphic functions on Y by properties (T1), (T2) (13.3.2) and (13.4.5)

proof: Recall (13.2.1) $\theta_\lambda = e^{-\frac{(\lambda|\lambda)}{2k}\delta}\sum_{\alpha\in M} e^{t_\alpha(\lambda)}$

(T1): (T2)
(13.3.2) $D(\theta_\lambda)=0$. (13.4.5) ___.
prop13.3. $\{\theta_\lambda \mid \lambda\in P_k \bmod (\quad)\}$ is a $\mathbb{C}$-basis of Th_k.
① $\theta_\lambda \subset \{$四个性质$\}$.

② by (T1) & (T2) ⇒ function $\subset \widetilde{Th} = \bigoplus_{k\geq 0}\widetilde{Th_k}$

by (13.3.2) ⇒ function $\subset Th$
by prop13.3 ⇒ function 写成 θ_λ 的线性组合.
by (13.4.5). ⇒ θ_λ. #.

§ 13.5.

Denote by $n = n(M)$ be the least positive integer such that $nM^* \subset M$ and $n(r|r) \in 2\mathbb{Z}$ for all $r \in M^*$.

Theorem 13.5. Let $\lambda \in P_k$. Then

(13.5.1) $$\Theta_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right) = (-i\tau)^{\frac{1}{2}l} \left|\frac{M^*}{kM}\right|^{-\frac{1}{2}} \times \sum_{\mu \in P_k \bmod (kM + \mathbb{C}\delta)} e^{-\frac{2\pi i}{k}(\bar{\mu}|\bar{\lambda})} \Theta_\mu(\tau, z, u)$$

(13.5.2) $$\Theta_\lambda(\tau+1, z, u) = e^{2\pi i|\lambda|^2/k} \Theta_\lambda(\tau, z, u)$$

Furthermore, if $A \in \Gamma(kn)$ (resp $\Gamma(kn) \cap \Gamma_\theta$) when M is even (resp. odd), then

(13.5.3) $$\Theta_\lambda|_{(A,j)} = v(A, j; k)\, \Theta_\lambda$$

where $v(A, j; k) \in \mathbb{C}$ and $|v(A, j; k)| = 1$.

Note that: $S \cdot (\tau, z, u) = \left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{(z|z)}{2\tau}\right)$.
$T \cdot (\tau, z, u) = (\tau+1, z, u)$.

proof: step 1: 简化证明条件.

Using that: for $g \in G = Mp_2(\mathbb{R}) \ltimes N$.
$$\underline{\Theta_\lambda|g} = \left(\underline{\Theta_{k\Lambda_0}|_{(0, k^{-1}\bar{\lambda}, 0)}}\right)\Big|_g = (\Theta_{k\Lambda_0}|g)\,|\, g^{-1}(0, -\tfrac{\bar{\lambda}}{k}, 0)\, g.$$
It suffices to prove the theorem for $\bar{\lambda} = 0$.

[by $\Theta_\lambda|_{(0,\alpha,0)} = \Theta_{\lambda - k\alpha}$ for $\alpha \in k^{-1}M^*$; (13.4.6)

$\Theta_{k\Lambda_0}|_{(0, -\frac{\bar{\lambda}}{k}, 0)} = \Theta_{k\Lambda_0 + \bar{\lambda}}$, $(k\Lambda_0 + \bar{\lambda} | \delta) = k$.

and by (13.2.1) $\Theta_{k\Lambda_0 + \bar{\lambda}} = e^{-\frac{(\bar{\lambda}|\bar{\lambda})}{2k}\delta} \sum_{\alpha \in M} e^{t_\alpha(k\Lambda_0 + \bar{\lambda})} = \Theta_\lambda$

Note $(0, 0, 0)v = v$. Then $\Theta_\lambda|g = \Theta_{k\Lambda_0}|g$.

· Note also that: $(\cdot|\cdot) \longrightarrow k(\cdot|\cdot)$.
$\Theta_\lambda(\tau, z, u)$ of degree k $\longrightarrow$ $\Theta_{k^{-1}\lambda}(\tau, z, ku)$ of degree 1
assume $\lambda = \Lambda_0$.

[$\Theta_\lambda(\tau, z, u) = e^{2\pi i k u} \sum e^{\pi i k(r|r) + 2\pi i k(r|z)} = \cdots\cdots$

$= \Theta_{k^{-1}\lambda}(\tau, z, ku)$.]

Step 2: 在上述条件下给出需要的结论.

By prop 13.4 b), we may write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (resp. $\in \Gamma_\theta$) if M is even (resp. odd).

(13.5.4) $\theta_{\lambda_0}|(A,j) = \sum_{\mu\in M^*\bmod M} f(\mu)\,\theta_{\lambda_0+\mu}$, where $f(\mu)\in\mathbb{C}$

- Fix $\alpha\in M^*$, since by (13.4.5), $\theta_{\lambda_0}|(\alpha,0,0) = e^{2\pi i(\alpha|\lambda_0)}\theta_{\lambda_0} = \theta_{\lambda_0}$

we get by (13.4.3):

$$\theta_{\lambda_0}|(A,j) = \theta_{\lambda_0}|(A,j)\,\underline{|(A,j)^{-1}(\alpha,0,0)(A,j)} = \underline{\theta_{\lambda_0}|(A,j)\,|(d\alpha,-c\alpha,0)}$$

- Hence applying $(A,j)^{-1}(\alpha,0,0)(A,j)$ to both siders of (13.5.4).

we get, $\theta_{\lambda_0}|(A,j) = \sum_{\mu\in M^*\bmod M} f(\mu)\,e^{\pi i(dc(\alpha|\alpha)+2d(\alpha|\mu))}\theta_{\lambda_0+\mu+c\alpha}$ (☆).

⌈ by (13.4.5), (13.4.6) we can get:

$$\theta_{\lambda_0}|(A,j) = \sum_{\mu\in M^*\bmod M} f(\mu)\,\underline{\theta_{\lambda_0+\mu}|(d\alpha,-c\alpha,0)}$$

Note $(d\alpha,-c\alpha,0) = (0,-c\alpha,0)(d\alpha,0,0)(0,0,-\pi i\,cd(\alpha|\alpha))$

$\Rightarrow e^{\pi i(dc(\alpha|\alpha)+2d(\alpha|\mu))}\theta_{\lambda_0+\mu+c\alpha}$ ⌋

Step 3. 开始证 (13.5.1).

- If $A=S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, then $\theta_{\lambda_0}|(S,j) \overset{(☆)}{=} \sum_{\mu\in M^*\bmod M} f(\mu)\theta_{\lambda_0+\mu+\alpha} \overset{(13.5.4)}{=} \sum_{\mu\in\cdots} f(\mu)\theta_{\lambda_0+\mu}$

$\Rightarrow f(\mu+\alpha) = f(\mu)$ for all $\alpha,\mu\in M^*$.

hence (13.5.5) $\theta_{\lambda_0}|(S,j) = v(S,j)\sum_{\mu\in M^*\bmod M}\theta_{\lambda_0+\mu}$ where $v(S,j)\in\mathbb{C}$.

- Note that $\begin{pmatrix}0&-1\\1&0\end{pmatrix}(\tau,z,u) = \left(-\frac{1}{\tau}, \frac{z}{\tau}, u-\frac{(z|z)}{2\tau}\right)$.

- Next we are going to compute this constant.

for any $\lambda\in M^*$, $\theta_{\lambda_0+\lambda} = \theta_{\lambda_0}|(0,-\lambda,0)$

so $\underline{\theta_{\lambda_0+\lambda}|(S,j)} = \theta_{\lambda_0}|(0,-\lambda,0)|(S,j) = \theta_{\lambda_0}|(S,j)|(S,j)^{-1}|(0,-\lambda,0)|(S,j)$

$= \theta_{\lambda_0}|(S,j)|\underline{S^{-1}(0,-\lambda,0)S} = \theta_{\lambda_0}|(S,j)|\underline{(-\lambda,0,0)}$

$= v(S,j)\sum_{\mu\in M^*\bmod M}\theta_{\lambda_0+\mu}|(-\lambda,0,0)$ by (13.5.5)

$= v(S,j)\sum_{\mu\in M^*(\bmod M)} e^{-2\pi i(\lambda|\mu)}\,\underline{\theta_{\lambda_0+\mu}}$ by (13.4.5)

$\in Th_1$.

$\Rightarrow$ The matrix of (S,j) w.r.t. it is

$$B = v(S,j)\left(e^{-2\pi i(\lambda|\mu)}\right)_{\lambda,\mu\in M^*\bmod M}$$

$(a_{ij})_{i,j\in I}$.

whose rows are pairwise orthogonal ? . Then.

$\underline{B\bar{B}^T} = \underline{|v(S,j)|^2\,|M^*/M|\,I}$ → identity matrix.

Note that by $(A,j)(A_1,j_1) = (AA_1,\ j(A_1\cdot\tau)j_1(\tau))$.

we have $(S,j)^2 = (-I_2,\ \underline{j(-\frac{1}{\tau})j(\tau)}) = (-I_2,\ i)$ since $j=\pm\sqrt{\tau}$.

Thus $(S,j)^8 = I$. $\Rightarrow$ $B^8 = I$

Hence $(\det B)^8 = 1$ $\Rightarrow$ $\underline{1 = |\det B|^2 = |\vartheta(S,j)|^2\,|M^*/M|}$.

$\Rightarrow$ $\underline{|\vartheta(S,j)| = |M^*/M|^{-\frac{1}{2}}}$ $\Rightarrow$ B is a unitary matrix.

Finally. $\Theta_{\lambda_0}|_{(S,j)}(i,0,0) = j(i)^{-\ell}\,\Theta_{\lambda_0}(S\cdot(i,0,0))$

$= j(i)^{-\ell}\,\Theta_{\lambda_0}(-\frac{1}{i},0,0) = j(i)^{-\ell}\,\Theta_{\lambda_0}(i,0,0)$

$\Theta_{\lambda_0+\mu}(i,0,0) = \sum_{\gamma\in M+\mu} e^{-\pi(\gamma|\gamma)} > 0$. for $\mu\in M^*$

Therefore by (13.5.5) $\Theta_{\lambda_0}|_{(S,j)}(i,0,0) = \vartheta(S,j)\sum\Theta_{\lambda_0+\mu}(i,0,0)$

$\Rightarrow$ $\vartheta(S,j) = j(i)^{-\ell}\,|M^*/M|^{-\frac{1}{2}} = (i)^{-\frac{\ell}{2}}\,|M^*/M|^{-\frac{1}{2}}$

So, $\Theta_{\lambda_0+\lambda}|_{(S,j)} = \underline{(-i)^{\frac{\ell}{2}}\,|M^*/M|^{-\frac{1}{2}}\sum e^{\cdots}}$

since $\Theta_{\lambda_0+\lambda}|_{(S,j)}(\tau,z,u) = j(\tau)^{-\ell}(\cdots) = \tau^{-\frac{\ell}{2}}(\cdots)$

$\Rightarrow$ $\underline{\Theta_{\lambda_0+\lambda}}|_{(S,j)}(\tau,z,u) = \underline{(-i\tau)^{\frac{\ell}{2}}\,|M^*/M|^{-\frac{1}{2}}\sum e^{\cdots}\,\Theta\ldots}$

#.

Step 4: 证 (13.5.2) 式.

since $T\cdot(\tau,z,u) = (\tau+1,z,u)$ where $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$.

Then $\Theta_{\lambda_0+\lambda}|_{(T,1)}(\tau,z,u) = \Theta_{\lambda_0+\lambda}(T(\tau,z,u))$

$= e^{2\pi i u}\sum_{\gamma\in M+\bar{\lambda}} e^{\pi i(\tau+1)(\gamma|\gamma)+2\pi i(\gamma|z)}$

$= e^{2\pi i u}\sum_{\gamma\in M+\bar{\lambda}} e^{\pi i\tau(\gamma|\gamma)+2\pi i(\gamma|z)+\underline{\pi i(\gamma|\gamma)}}$

Note that $(\alpha+\lambda|\alpha+\lambda) = \underline{\underbrace{(\alpha|\alpha)}_{2\mathbb{Z}}+\underbrace{2(\alpha|\lambda)}_{2\mathbb{Z}}}+(\lambda|\lambda)$

Then $e^{\pi i(\gamma|\gamma)} = e^{\pi i(\lambda|\lambda)}$ $\Leftarrow$ m is even.

Now $\Theta_{\lambda_0+\lambda}|_{(T,1)}(\tau,z,u) = e^{\pi i|\lambda|^2}\,\Theta_{\lambda_0+\lambda}(\tau,z,u)$

得 (13.5.2) 式

Rem: since S and T^2 generate Γ_θ.

同理考虑 $\tau=2$, when m is odd.

then $\Theta_{\lambda_0+\lambda}|_{(T^2,1)} = e^{2\pi i|\lambda|^2}\,\Theta_{\lambda_0+\lambda}$.

第五步：证 (13.5.3)

let: $A \in \Gamma(n)$ when m is even or $A \in \Gamma(n) \cap \Gamma_\theta$ when m is odd.

write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $a \equiv d \equiv 1 \bmod n$, $b \equiv c \equiv 0 \bmod n$.

by (A)

$$\theta_{\lambda_0}|_{(A,j)} = \sum_{\mu \in M^* \bmod M} f(\mu) e^{\pi i (dc(\alpha|\alpha) + 2d(\alpha|\mu))} \theta_{\lambda_0 + \mu + c\alpha} \quad (\text{A}).$$

let $d = 1 + d_1 n$, $c = c_1 n$ where $c_1, d_1 \in \mathbb{Z}$.

Then $dc(\alpha|\alpha) + 2d(\alpha|\mu) = c_1 d n(\alpha|\alpha) + 2(\alpha|\mu) + 2d_1 n(\alpha|\mu)$

$\equiv 2(\alpha|\mu) \pmod{2\mathbb{Z}}$. $n(\alpha|\alpha) \in 2\mathbb{Z}$, & $nM^* \subset M$. $\Rightarrow (n\alpha|\mu) \in \mathbb{Z}$

Hence $f(\mu) = f(\mu) e^{2\pi i (\alpha|\mu)}$

$\Rightarrow f(\mu) = 0$ if $\mu \notin M$. $\Rightarrow \theta_{\lambda_0}|_{(A,j)} = v(A,j)\, \theta_{\lambda_0}$. by (A).

since $c\alpha = c_1 n\alpha \in 1 \cdot M$. $\Rightarrow \theta_{\lambda_0 + \mu + c\alpha} = \theta_{\lambda_0}$

where $v(A,j) \in \mathbb{C}$.

The fact $|v(A,j)| = 1$ follows from Cor 13.5 below.

#.

Cor 13.5: The matrix of a transformation from $M_{p_v}(\mathbb{Z})$ (resp. $M^\theta_{p_v}(\mathbb{Z})$) if m is even (resp. odd) in the basis $\{\theta_\lambda\}_{\lambda \dots}$ is unitary.

Examp 13.5.

$$\theta_{n,m}\left(-\frac{1}{\tau}, \frac{z}{\tau}, u - \frac{z^2}{2\tau}\right)$$
$$= (-i\tau)^{\frac{1}{2}} (2m)^{-\frac{1}{2}} \sum_{n' \in \mathbb{Z} \bmod 2m\mathbb{Z}} e^{-\frac{\pi i n n'}{m}} \theta_{n',m}(\tau, z, u).$$

$(\alpha|\alpha) = 2$. so $2(\mu|\alpha) = nn'$.